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TRIPLY CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS.

By Luther Pfahler Eisenhart.

1. When the Cartesian coördinates x_1, x_2, x_3 of a point M in space are functions of three parameters u_1, u_2, u_3 which are functionally independent and satisfy three equations of the form

$$\begin{aligned}\frac{\partial^2\theta}{\partial u_1\partial u_2} &= b_{12}\frac{\partial\theta}{\partial u_1} + b_{21}\frac{\partial\theta}{\partial u_2},\\ \frac{\partial^2\theta}{\partial u_2\partial u_3} &= b_{23}\frac{\partial\theta}{\partial u_2} + b_{32}\frac{\partial\theta}{\partial u_3},\\ \frac{\partial^2\theta}{\partial u_3\partial u_1} &= b_{31}\frac{\partial\theta}{\partial u_3} + b_{13}\frac{\partial\theta}{\partial u_1},\end{aligned}\tag{1}$$

where the functions b_{ij} ordinarily are functions of u_1, u_2, u_3, the parametric surfaces $u_i =$ const. ($i = 1, 2, 3$) cut along conjugate systems of curves. Such a system is called a *triply conjugate system.*

The functions b_{ij} are not arbitrary, but are subject to certain conditions, which we shall now determine. When the two expressions for $\partial^3\theta/\partial u_1\partial u_2\partial u_3$ obtained by differentiating the first two of equations (1) are equated, we get

$$\frac{\partial\theta}{\partial u_1}\left(\frac{\partial b_{12}}{\partial u_3} + b_{12}b_{13} - b_{12}b_{23} - b_{32}b_{13}\right) + \frac{\partial\theta}{\partial u_2}\left(\frac{\partial b_{21}}{\partial u_3} - \frac{\partial b_{23}}{\partial u_1}\right)$$
$$- \frac{\partial\theta}{\partial u_3}\left(\frac{\partial b_{32}}{\partial u_1} + b_{32}b_{31} - b_{12}b_{31} - b_{21}b_{13}\right) = 0.$$

Since, by hypothesis, this condition must be satisfied by three functionally independent solutions of (1), the expressions in parentheses must be equal to zero. Proceeding in like manner with the first and third of equations (1) and also with the second and third, we obtain similar conditions all of which may be written:

$$\frac{\partial b_{ij}}{\partial u_k} = \frac{\partial b_{ik}}{\partial u_j},\tag{2}$$

$$\frac{\partial b_{ij}}{\partial u_k} + b_{ij}b_{ik} - b_{ij}b_{jk} - b_{kj}b_{ik} = 0,\tag{3}$$

where the subscripts i, j, k are different from one another and take on the values 1, 2, 3.

The first three of these equations are satisfied, if we put

$$b_{ij} = \frac{\partial}{\partial u_j} \log a_i, \tag{4}$$

thus defining three functions a_i, to within a multiple which is a function of u_i alone. When these expressions are substituted in (3), the latter reduce to the three equations

$$\frac{\partial^2 a_i}{\partial u_j \partial u_k} = \frac{\partial \log a_j}{\partial u_k} \frac{\partial a_i}{\partial u_j} + \frac{\partial \log a_k}{\partial u_j} \frac{\partial a_i}{\partial u_k}. \tag{5}$$

Now equations (1) become

$$\frac{\partial^2 \theta}{\partial u_i \partial u_j} = \frac{\partial \log a_i}{\partial u_j} \frac{\partial \theta}{\partial u_i} + \frac{\partial \log a_j}{\partial u_i} \frac{\partial \theta}{\partial u_j} \tag{6}$$

From the preceding discussion it follows that, when any three functions a_i satisfy (5), the corresponding equations (6) are compatible and admit solutions not functionally expressible in terms of two of them. The general solution involves three arbitrary functions of a single variable.*

2. We are concerned in this paper with the triply conjugate systems with equal invariants, that is those for which the functions a_i satisfy the conditions

$$\frac{\partial^2 \log a_i}{\partial u_i \partial u_j} = \frac{\partial^2 \log a_j}{\partial u_i \partial u_j}.$$

These equations may be replaced by

$$\frac{a_1}{a_2} = \frac{\varphi_2}{\psi_1}, \quad \frac{a_2}{a_3} = \frac{\varphi_3}{\psi_2}, \quad \frac{a_3}{a_1} = \frac{\varphi_1}{\psi_3}, \tag{7}$$

where a function φ_i or ψ_i is independent of u_i. Multiplying these equations together, we get

$$\frac{\varphi_1}{\psi_1} \cdot \frac{\varphi_2}{\psi_2} \cdot \frac{\varphi_3}{\psi_3} = 1.$$

If we take the logarithmic derivative of this equation with respect to u_2 and u_3, we obtain

$$\frac{\partial^2}{\partial u_2 \partial u_3} \log \frac{\varphi_1}{\psi_1} = 0,$$

that is φ_1/ψ_1 is equal to the ratio of a function of u_3 alone and a function of u_2 alone. Since similar results are true for φ_2/ψ_2 and φ_3/ψ_3, it is readily shown that in all generality the above equation may be replaced by

$$\frac{\varphi_1}{\psi_1} = \frac{U_3}{U_2}, \quad \frac{\varphi_2}{\psi_2} = \frac{U_1}{U_3}, \quad \frac{\varphi_3}{\psi_3} = \frac{U_2}{U_1},$$

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* Darboux, Lecons sur la théorie générale des surfaces, vol. 4, p. 271.

where U_i denotes a function of u_i alone. When these values are substituted in (7) and we put

$$e^{\sigma_1} = \varphi_1 U_2, \qquad e^{\sigma_2} = \varphi_2 U_3, \qquad e^{\sigma_3} = \varphi_3 U_1,$$

we may replace (7) by

$$(8) \qquad a_1 = \frac{e^{\sigma_2+\sigma_3}}{\mu}, \qquad a_2 = \frac{e^{\sigma_3+\sigma_1}}{\mu}, \qquad a_3 = \frac{e^{\sigma_1+\sigma_2}}{\mu},$$

where μ is in general a function of all three parameters and σ_i is independent of u_i.

When the expressions (8) are substituted in (5), we find that μ must satisfy the three equations

$$(9) \qquad \frac{\partial^2\varphi}{\partial u_i \partial u_j} = \frac{\partial\sigma_k}{\partial u_j}\frac{\partial\varphi}{\partial u_i} + \frac{\partial\sigma_k}{\partial u_i}\frac{\partial\varphi}{\partial u_j} + \left(\frac{\partial\sigma_j}{\partial u_i}\frac{\partial\sigma_i}{\partial u_j} - \frac{\partial\sigma_k}{\partial u_j}\frac{\partial\sigma_j}{\partial u_i} - \frac{\partial\sigma_k}{\partial u_i}\frac{\partial\sigma_i}{\partial u_j}\right)\varphi.$$

If the expressions (8) are substituted in (6), and at the same time we put

$$(10) \qquad \theta = \varphi/\mu,$$

the equations for φ reduce to (9). Hence the problem of finding triply conjugate systems with equal invariants reduces to the integration of equations (9). In fact, if θ_1, θ_2, θ_3, θ_4 are four independent solutions of (9), then

$$(11) \qquad x_1 = \frac{\theta_1}{\theta_4}, \qquad x_2 = \frac{\theta_2}{\theta_4}, \qquad x_3 = \frac{\theta_3}{\theta_4},$$

are the Cartesian coördinates of space referred to such a triply conjugate system with equal point invariants.

3. We consider now the conditions which the three functions σ_i must satisfy in order that (9) may admit four functionally independent solutions.

In the first place we observe that if we have three functions a_i given by (8) which satisfy the conditions of the problem, the same system of equations is given by the functions σ_i' and μ', defined by

$$(12) \qquad \sigma_i' = \sigma_i + U_j + U_k, \qquad \mu' = \mu e^{U_1+U_2+U_3},$$

where U_i is a function of u_i alone. In fact,

$$a_i' = \frac{e^{\sigma_j'+\sigma_k'}}{\mu'} = e^{U_i} a_i.$$

But in § 1 we observed that a_i was determined only to within a factor of u_i alone. This establishes the result.

We turn now to the determination of the conditions which the functions σ_i must satisfy in order that equations (9) may be consistent. When we equate in pairs the expressions for $\partial^3\varphi/\partial u_1\partial u_2\partial u_3$ obtained from (9) by differentiation, we get the three conditions

$$A - 2B + C = 0, \qquad A + B - 2C = 0, \qquad B + C - 2A = 0,$$

where we have put

$$A = \left(\frac{\partial \sigma_2}{\partial u_1} - \frac{\partial \sigma_3}{\partial u_1}\right)\frac{\partial^2 \sigma_1}{\partial u_2 \partial u_3}, \qquad B = \left(\frac{\partial \sigma_3}{\partial u_2} - \frac{\partial \sigma_1}{\partial u_2}\right)\frac{\partial^2 \sigma_2}{\partial u_3 \partial u_1},$$

$$C = \left(\frac{\partial \sigma_1}{\partial u_3} - \frac{\partial \sigma_2}{\partial u_3}\right)\frac{\partial^2 \sigma_3}{\partial u_1 \partial u_2}.$$

In order that these three conditions may be satisfied simultaneously, we must have

$$(13) \qquad A = B = C.$$

We consider first the case

$$(14) \qquad A = B = C = 0.$$

If

$$\frac{\partial \sigma_2}{\partial u_1} - \frac{\partial \sigma_3}{\partial u_1} = 0,$$

we have on differentiating this equation separately with respect to u_2 and u_3

$$\frac{\partial^2 \sigma_2}{\partial u_1 \partial u_3} = 0, \qquad \frac{\partial^2 \sigma_3}{\partial u_1 \partial u_2} = 0,$$

in which case B and C are zero also. In order that these three equations be satisfied we must have

$$\sigma_2 = U_1 + U_3, \qquad \sigma_3 = U_1 + U_2,$$

where U_i is an arbitrary function of u_i alone. In consequence of (12), we may take $\sigma_2 = \sigma_3 = 0$, and σ_1 arbitrary. Hence a first solution is afforded by the three values

$$(15) \qquad \sigma_i \text{ arbitrary}, \qquad \sigma_j = \sigma_k = 0.$$

We consider secondly the other possibility for (14) afforded by

$$\frac{\partial^2 \sigma_1}{\partial u_2 \partial u_3} = 0, \qquad \frac{\partial^2 \sigma_2}{\partial u_3 \partial u_1} = 0, \qquad \frac{\partial^2 \sigma_3}{\partial u_1 \partial u_2} = 0.$$

Because of (12) the general solution in this case is

$$(16) \qquad \sigma_1 = U_2 - U_3, \qquad \sigma_2 = U_3 - U_1, \qquad \sigma_3 = U_1 - U_2,$$

where U_i is an arbitrary function of u_i alone, none of which may be constant; otherwise we have a type reducible to (15).

We consider finally the general case where the common value of A,

B and C is different from zero. If we put

$$\frac{\partial^2\sigma_1}{\partial u_2\partial u_3} = \frac{1}{h}\left(\frac{\partial\sigma_1}{\partial u_2} - \frac{\partial\sigma_3}{\partial u_2}\right)\left(\frac{\partial\sigma_2}{\partial u_3} - \frac{\partial\sigma_1}{\partial u_3}\right), \tag{17}$$

where h is to be determined, equations (14) may be replaced by

$$\begin{aligned} \frac{\partial^2\sigma_2}{\partial u_3\partial u_1} &= \frac{1}{h}\left(\frac{\partial\sigma_2}{\partial u_3} - \frac{\partial\sigma_1}{\partial u_3}\right)\left(\frac{\partial\sigma_3}{\partial u_1} - \frac{\partial\sigma_2}{\partial u_1}\right), \\ \frac{\partial^2\sigma_3}{\partial u_1\partial u_2} &= \frac{1}{h}\left(\frac{\partial\sigma_3}{\partial u_1} - \frac{\partial\sigma_2}{\partial u_1}\right)\left(\frac{\partial\sigma_1}{\partial u_2} - \frac{\partial\sigma_3}{\partial u_2}\right). \end{aligned} \tag{18}$$

When equations (17) and (18) are differentiated with respect to u_1, u_2 and u_3 respectively, it is found that h is a constant.

If in equation (17) we make the substitution

$$\sigma_1 = \sigma_2 + \sigma_3 + \tau,$$

the resulting equation is

$$h\frac{\partial^2\tau}{\partial u_2\partial u_3} + \frac{\partial\tau}{\partial u_2}\frac{\partial\tau}{\partial u_3} = 0,$$

of which the general integral is $\tau = h \log(\varphi_2 + \varphi_3)$, where φ_2 and φ_3 are arbitrary functions independent of u_2 and u_3 respectively. When this value of τ is put in the above equation for σ_1, we may in all generality give u_1 a constant value in the left-hand member, in which case we have

$$\sigma_1 = \overline{U}_2 + \overline{U}_3 + h \log(U_2 - U_3),$$

where $\overline{U}_i$ and U_i are functions of u_i alone.

When this value is substituted in (17), the resulting equation may be put in the form

$$\frac{hU_2'}{\dfrac{\partial\sigma_3}{\partial u_2} - \overline{U}_2'} - U_2 = \frac{hU_3'}{\dfrac{\partial\sigma_2}{\partial u_3} - \overline{U}_3'} - U_3.$$

Since the respective members of this equation are independent of u_3 and u_2, they are equal to a function of u_1 alone, say $-U_1$. Then we have by integration

$$\begin{aligned} \sigma_2 &= \overline{U}_3 + h\log(U_3 - U_1) + \overline{U}_1, \\ \sigma_3 &= \overline{U}_2 + h\log(U_1 - U_2) + \psi(u_1), \end{aligned}$$

where $\overline{U}_1$ and ψ are functions of u_1 alone.

We remark that none of the functions U_1, U_2, U_3 may be constant. For if $U_1 = \text{const.}$, we have

$$\frac{\partial^2\sigma_2}{\partial u_1\partial u_3} = \frac{\partial^2\sigma_3}{\partial u_1\partial u_2} = 0,$$

which is contrary to hypothesis.

When the above values of σ_1, σ_2 and σ_3 are substituted in the first of (18), we get $\psi' = \overline{U}_1'$. If we write $\psi = \overline{U}_1 + 2a$, and then replace $\overline{U}_1$, $\overline{U}_2$, $\overline{U}_3$ by $\overline{U}_1 - a$, $\overline{U}_2 - a$, $\overline{U}_3 + a$ respectively, we get the same result as if we had taken $a = 0$; this therefore will be done. In view of this result and (12) we have in all generality

$$(19)\quad \sigma_1 = h \log (U_2 - U_3), \quad \sigma_2 = h \log (U_3 - U_1), \quad \sigma_3 = h \log (U_1 - U_2),$$

where none of the functions U_i is constant.*

4. We consider first the case given by (15). Equations (9) reduce to

$$(20)\quad \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} = 0, \quad \frac{\partial^2 \varphi}{\partial u_3 \partial u_1} = 0, \quad \frac{\partial^2 \varphi}{\partial u_2 \partial u_3} = \frac{\partial \sigma_1}{\partial u_3}\frac{\partial \varphi}{\partial u_2} + \frac{\partial \sigma_1}{\partial u_2}\frac{\partial \varphi}{\partial u_3}.$$

From the first two of these equations it follows that φ is the sum of a function of u_1 alone and a function independent of u_1. Hence the point coördinates of a triply conjugate system of this type are of the form

$$x = \frac{\varphi_1 + U_1}{\overline{\varphi}_1 + \overline{U}_1},$$

where φ_1 and $\overline{\varphi}_1$ are solutions of the third of equations (20) and are independent of u_1.

When in particular $\sigma_1 = 0$, we have

$$x = \frac{U_1 + U_2 + U_3}{\overline{U}_1 + \overline{U}_2 + \overline{U}_3},$$

where U_i and $\overline{U}_i$ are functions of u_i alone.

5. When the values (16) are substituted in equations (9), it is found that by a transformation of variables U_i may be replaced by u_i in all generality for $i = 1, 2, 3$, and the equations become

$$(21)\quad \begin{aligned} \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \varphi}{\partial u_1} - \frac{\partial \varphi}{\partial u_2} + 3\varphi &= 0, \\ \frac{\partial^2 \varphi}{\partial u_2 \partial u_3} + \frac{\partial \varphi}{\partial u_2} - \frac{\partial \varphi}{\partial u_3} + 3\varphi &= 0, \\ \frac{\partial^2 \varphi}{\partial u_3 \partial u_1} + \frac{\partial \varphi}{\partial u_3} - \frac{\partial \varphi}{\partial u_1} + 3\varphi &= 0. \end{aligned}$$

We observe that if φ is any solution of this system so also is any of the derivatives of φ.

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* Cf. Darboux, Lecons sur les systémes orthogonaux, second edition, pp. 231, et seq. In these pages Darboux solves the problem of finding all triply orthogonal systems such that all the surfaces are isothermic, that is, the lines of curvature on each surface form an isothermic net. These systems are triply conjugate with equal point invariants.

Gronwall in the next paper* has found the solution $\varphi(u_1, u_2, u_3)$ of these equations determined uniquely by its values $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$. If f_0 and f_1 denote Bessel's functions of order zero and one respectively, the integral is

$$\begin{aligned}\varphi(u_1, u_2, u_3) = {} & e^{-u_2}\varphi(u_1, 0, u_3) + e^{u_1}\varphi(0, u_2, u_3) - e^{u_1-u_2}f_0(u_1u_2)\varphi(0, 0, u_3) \\ & - e^{-u_2}\int_0^{u_1} e^{u_1-v_1}4u_2f_1(u_2(u_1 - v_1))\varphi(v_1, 0, u_3)dv_1 \\ & - e^{u_1}\int_0^{u_2} e^{v_2-u_2}4u_1f_1(u_1(u_2 - v_2))\varphi(0, v_2, u_3)dv_2,\end{aligned}$$

where $\varphi(u_1, 0, u_3)$ and $\varphi(0, u_2, u_3)$ are given by the similar equations, namely

$$\begin{aligned}\varphi(0, u_2, u_3) = {} & e^{-u_3}\varphi(0, u_2, 0) + e^{u_2}\varphi(0, 0, u_3) - e^{u_2-u_3}f_0(u_2u_3)\varphi(0, 0, 0) \\ & - e^{-u_3}\int_0^{u_2} e^{u_2-v_2}4u_3f_1(u_3(u_2 - v_2))\varphi(0, v_2, 0)dv_2 \\ & - e^{u_2}\int_0^{u_3} e^{v_3-u_3}4u_2f_1(u_2(u_3 - v_3))\varphi(0, 0, v_3)dv_3,\end{aligned}$$

and

$$\begin{aligned}\varphi(u_1, 0, u_3) = {} & e^{-u_1}\varphi(0, 0, u_3) + e^{u_3}\varphi(u_1, 0, 0) - e^{u_3-u_1}f_0(u_3u_1)\varphi(0, 0, 0) \\ & - e^{-u_1}\int_0^{u_3} e^{u_3-v_3}4u_1f_1(u_1(u_3 - v_3))\varphi(0, 0, v_3)dv_3 \\ & - e^{u_3}\int_0^{u_1} e^{v_1-u_1}4u_3f_1(u_3(u_1 - v_1))\varphi(v_1, 0, 0)dv_1.\end{aligned}$$

Gronwall† reduces this solution to other forms.

6. When the values (19) are substituted in equations (9), we find that by a transformation of the variables we may in all generality take $U_i = u_i$ $(i = 1, 2, 3)$. Then the equations become

$$(22)\qquad \begin{aligned}\frac{\partial^2\varphi}{\partial u_1\partial u_2} &= \frac{h}{u_2 - u_1}\left(\frac{\partial\varphi}{\partial u_1} - \frac{\partial\varphi}{\partial u_2}\right), \\ \frac{\partial^2\varphi}{\partial u_2\partial u_3} &= \frac{h}{u_3 - u_2}\left(\frac{\partial\varphi}{\partial u_2} - \frac{\partial\varphi}{\partial u_3}\right), \\ \frac{\partial^2\varphi}{\partial u_3\partial u_1} &= \frac{h}{u_1 - u_3}\left(\frac{\partial\varphi}{\partial u_3} - \frac{\partial\varphi}{\partial u_1}\right).\end{aligned}$$

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* On a system of linear partial differential equations of the hyperbolic type, pp. 273–276, of this number of the Annals.

† L. c.

We make the following observation concerning these equations. If each be differentiated with respect to u_1, u_2 and u_3, and we put

$$\varphi_1 = \frac{\partial^3\varphi}{\partial u_1 \partial u_2 \partial u_3}$$

we find that φ_1 satisfies the equations

$$\begin{aligned} &\frac{\partial^2\varphi_1}{\partial u_1\partial u_2} = \frac{h-1}{u_2-u_1}\left(\frac{\partial\varphi_1}{\partial u_1} - \frac{\partial\varphi_1}{\partial u_2}\right), \\ (23) \qquad &\frac{\partial^2\varphi_1}{\partial u_2\partial u_3} = \frac{h-1}{u_3-u_2}\left(\frac{\partial\varphi_1}{\partial u_2} - \frac{\partial\varphi_1}{\partial u_3}\right), \\ &\frac{\partial^2\varphi_1}{\partial u_3\partial u_1} = \frac{h-1}{u_1-u_3}\left(\frac{\partial\varphi_1}{\partial u_3} - \frac{\partial\varphi_1}{\partial u_1}\right). \end{aligned}$$

Hence when the general integral of the system (22) is known for a given value of h, we can find the integrals of the equations for $h - n$, where n is an integer.

Again if we put

$$\varphi = (u_1 - u_2)^h \overline{\varphi}$$

in the first of equations (23), it becomes

$$\frac{\partial^2\overline{\varphi}}{\partial u_1\partial u_2} = \frac{-h(h+1)}{(u_1-u_2)^2}\overline{\varphi}.$$

From this it is seen that, when $h = -1$, the general integral of the first of equations (22) is $(u_1 - u_2)^h(\psi_1 + \psi_2)$, where ψ_1 and ψ_2 are independent of u_1 and u_2 respectively. When it is required that this expression satisfy the second and third of (22), it is found that the general integral of this system of equations for $h = -1$ is of the form

$$\varphi = \frac{U_1}{(u_1-u_2)(u_1-u_3)} + \frac{U_2}{(u_2-u_3)(u_2-u_1)} + \frac{U_3}{(u_3-u_1)(u_3-u_2)},$$

where U_i is an arbitrary function of u_i alone. In view of the preceding remarks we may obtain from this expression by differentiation the general integral of (22) for *any negative integral* value of h.

If in equations (6) we put

$$\begin{aligned} &a_1 = (u_1 - u_2)^{m_2}(u_3 - u_1)^{m_3}, \\ (24) \qquad &a_2 = (u_2 - u_3)^{m_3}(u_1 - u_2)^{m_1}, \\ &a_3 = (u_3 - u_1)^{m_1}(u_2 - u_3)^{m_2}, \end{aligned}$$

we get

$$(25) \qquad \frac{\partial^2\varphi}{\partial u_i\partial u_j} = -\frac{m_j}{u_i-u_j}\frac{\partial\theta}{\partial u_i} + \frac{m_i}{u_i-u_j}\frac{\partial\theta}{\partial u_j}.$$

When the transformation of Laplace, defined by

$$\varphi_1 = \varphi - \frac{a_2}{\frac{\partial a_2}{\partial u_1}} \frac{\partial \varphi}{\partial u_1}, \tag{26}$$

is applied to this system of equations, we obtain another system of the form (6) in which the quantities $\bar{a}_i$, entering in the coefficients, are of the form

$$\begin{aligned} \bar{a}_1 &= (u_1 - u_2)^{m_2+1}(u_3 - u_1)^{m_3}, \\ \bar{a}_2 &= (u_2 - u_3)^{m_3}(u_1 - u_2)^{m_1-1}, \\ \bar{a}_3 &= (u_3 - u_1)^{m_1-1}(u_2 - u_3)^{m_2+1}, \end{aligned}$$

which are of the same form as (24) with m_1, m_2, m_3 replaced by $m_1 - 1$, $m_2 + 1$, m_3 respectively.

Evidently equations (22) are of the form (25) with $m_1 = m_2 = m_3 = h$. Hence if h is a positive integer and we apply the transformation (26) h times in succession, we get the system

$$\begin{aligned} &\frac{\partial^2 \psi}{\partial u_1 \partial u_2} = -\frac{2h}{u_1 - u_2}\frac{\partial \psi}{\partial u_1}, \qquad \frac{\partial^2 \psi}{\partial u_3 \partial u_1} = \frac{h}{u_3 - u_1}\frac{\partial \psi}{\partial u_1}, \\ &\frac{\partial^2 \psi}{\partial u_2 \partial u_3} = -\frac{h}{u_2 - u_3}\frac{\partial \psi}{\partial u_2} + \frac{2h}{u_2 - u_3}\frac{\partial \psi}{\partial u_3}. \end{aligned} \tag{27}$$

The general integral of the first two of these equations is

$$\psi = \int (u_1 - u_2)^{2h}(u_3 - u_1)^h U_1 du_1 + \rho_1, \tag{28}$$

where U_1 is an arbitrary function of u_1 alone, and ρ_1 is independent for u_1. When this expression for ψ is substituted in the third of (27), we find that ρ_1 must be a solution of the third. But the general integral of this equation is*

$$\rho_1 = (u_2 - u_3)^{3h+1}\frac{\partial^{3h}}{\partial u_2^{2h}\partial u_3^{h}}\left(\frac{U_2 - U_3}{u_2 - u_3}\right),$$

where U_2 and U_3 are arbitrary functions of u_2 and u_3 respectively. Hence we have the general solution of (27).

Equation (26) may be written

$$\varphi = \varphi_1 - \frac{\bar{a}_1}{\frac{\partial \bar{a}_1}{\partial u_2}}\frac{\partial \varphi_1}{\partial u_2},$$

which is the inverse of the Laplace transformation. For the present case this and the inverses of the successive transformations become

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* Darboux, Lecons, vol. 2, p. 65.

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\begin{aligned}
\varphi &= \varphi_1 + \frac{u_1 - u_2}{h+1}\frac{\partial \varphi_1}{\partial u_2}, \\
\varphi_1 &= \varphi_2 + \frac{u_1 - u_2}{h+2}\frac{\partial \varphi_2}{\partial u_2}, \\
&\cdots\cdots\cdots \\
\varphi_{h-1} &= \varphi_h + \frac{u_1 - u_2}{2h}\frac{\partial \varphi_h}{\partial u_2}.
\end{aligned} \tag{29}
$$

Now φ_h is the function ψ given by (28), and hence φ is readily obtained by the differentiations indicated in (29).

7. Consider now the case when $h > 0$. The definite integral

$$\int_a^{u_1} \psi(t)(t - u_1)^h(u_2 - t)^h(u_3 - t)^h dt, \tag{30}$$

where ψ is an arbitrary function of t and a is a constant, is an evident generalization of the integral of the first of equations (22) given by Poisson.*

By applying the usual method of differentiating a definite integral we show that the integral (30) is a solution of equations (22) whatever be ψ.

The same is true of the integrals

$$\int_a^{u_2} \psi(t)(u_1 - t)^h(t - u_2)^h(u_3 - t)^h dt,$$

$$\int_a^{u_3} \psi(t)(u_1 - t)^h(u_2 - t)^h(t - u_3)^h dt.$$

Thus we have three independent integrals each involving an arbitrary function.

8. For the case $h < 0$ we apply to the system (22) the method of Riemann, as Gronwall has done in the case of equations (21)†, and find to within quadratures the unique integral $\varphi(u_1, u_2, u_3)$ determined by arbitrary values of $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$.

We recall that to integrate an equation

$$\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + a(u_1, u_2)\frac{\partial \varphi}{\partial u_1} + b(u_1, u_2)\frac{\partial \varphi}{\partial u_2} + c(u_1, u_2)\varphi = 0$$

by the Riemann method we form the adjoint equation

$$\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{\partial}{\partial v_1}(a(v_1, v_2)t) - \frac{\partial}{\partial v_2}(b(v_1, v_2)t) + c(v_1, v_2)t = 0,$$

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* Journal de l'Ecole Polytechnique, Cahier 19 (1823), p. 215.

† L. c.

and find a solution $t(v_1, v_2;\ u_1, u_2)$ satisfying the initial conditions

$$\begin{aligned} t(v_1, u_2;\ u_1, u_2) &= e^{\int_{u_1}^{v_1} b(v_1,\, u_2)dv_1}, \\ t(u_1, v_2;\ u_1, u_2) &= e^{\int_{u_2}^{v_2} a(u_1,\, v_2)dv_2}. \end{aligned} \tag{31}$$

Then the integral of the given equation reducing to given values of $\varphi(0, u_2)$ and $\varphi(u_1, 0)$ is

$$\begin{aligned} \varphi(u_1, u_2) = t(u_1, 0;\ u_1, u_2)\varphi(u_1, 0) + t(0, u_2;\ u_1, u_2)\varphi(0, u_2) \\ - t(0, 0;\ u_1, u_2)\varphi(0, 0) \\ + \int_0^{u_1}\left[b(v_1, 0)t(v_1, 0;\ u_1, u_2) - \frac{\partial}{\partial v_1}t(v_1, 0;\ u_1, u_2)\right]\varphi(v_1, 0)dv_1 \\ + \int_0^{u_2}\left[a(0, v_2)t(0, v_2;\ u_1, u_2) - \frac{\partial}{\partial v_2}t(0, v_2;\ u_1, u_2)\right]\varphi(0, v_2)dv_2. \end{aligned} \tag{32}$$

9. For the first of equations (22)

$$a(u_1, u_2) = \frac{h}{u_1 - u_2}, \qquad b(u_1, u_2) = \frac{h}{u_2 - u_1},$$

and the adjoint equation is

$$\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{h}{v_1 - v_2}\frac{\partial t}{\partial v_1} + \frac{h}{v_1 - v_2}\frac{\partial t}{\partial v_2} + \frac{2h}{(v_1 - v_2)^2}t = 0.$$

In this case the initial values (31) are reducible to

$$t(v_1, u_2;\ u_1, u_2) = \left(\frac{u_1 - u_2}{v_1 - u_2}\right)^h, \qquad t(u_1, v_2;\ u_1, u_2) = \left(\frac{u_2 - u_1}{v_2 - u_1}\right)^h.$$

Darboux* has shown that the solution of the adjoint equation satisfying these conditions is

$$\begin{aligned} t(v_1, v_2;\ u_1, u_2) = (u_2 - v_1)^h(v_2 - v_1)^{-2h}(v_2 - u_1)^h \\ \times F\left(-h, -h, 1, \frac{(v_1 - u_1)\cdot(v_2 - u_2)}{(v_1 - u_2)\cdot(v_2 - u_1)}\right), \end{aligned}$$

where F denotes the hypergeometric series with the constants $-h$, $-h$, 1 and the argument $(v_1 - u_1)(v_2 - u_2)/(v_1 - u_2)(v_2 - u_1)$.

In order to obtain the expression (32), we note that

$$t(u_1, 0;\ u_1, u_2) = u_1^{-h}(u_1 - u_2)^h, \qquad t(0, u_2;\ u_1, u_2) = u_2^{-h}(u_2 - u_1)^h,$$
$$t(0, 0;\ u_1, u_2) = 0,$$

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[*] Lecons, 2d ed., vol. 2, p. 83.

and that $F(-h, -h, 1, 1)$ is convergent. If we put, for the sake of brevity,

$$\Phi(v_1, v_2;\ u_1, u_2) = hu_1^h u_2 v_1^{-(2h+1)}(v_1 - u_2)^{h-1} F\left(-h, -h, 1, \frac{u_2}{u_1}\frac{v_1 - u_1}{v_1 - u_2}\right) + u_1^{h-1}u_2(u_2 - u_1)v_1^{-2h}(v_1 - u_2)^{h-2}F',$$

where F' denotes the derivative of F with respect to the argument, we have

$$\begin{aligned}\varphi(u_1, u_2, u_3) &= u_1^{-h}(u_1 - u_2)^h\varphi(u_1, 0, u_3) + u_2^{-h}(u_2 - u_1)^h\varphi(0, u_2, u_3)\\ &\quad + \int_0^{u_1}\Phi(v_1, v_2;\ u_1, u_2)\varphi(v_1, 0, u_3)dv_1\\ &\quad + \int_0^{u_2}\Phi(v_2, v_1;\ u_2, u_1)\varphi(0, v_2, u_3)dv_2.\end{aligned} \tag{33}$$

Since the right-hand member of this equation reduces to $\varphi(0, u_2, u_3)$ for $u_1 = 0$, and since the second of equations (22) is independent of u_1, it follows that the expression (33) will satisfy the second equation, if we take for $\varphi(0, u_2, u_3)$ the solution of this second equation which for $u_2 = 0$ and $u_3 = 0$ takes the respective given values $\varphi(0, 0, u_3)$ and $\varphi(0, u_2, 0)$. In like manner the expression (33) will satisfy the third of (22), if we take for $\varphi(u_1, 0, u_3)$ that solution of this equation reducing for $u_1 = 0$ and $u_3 = 0$ to $\varphi(0, 0, u_3)$ and $\varphi(u_1, 0, 0)$ respectively.

In order to obtain these desired solutions $\varphi(0, u_2, u_3)$ and $\varphi(u_1, 0, u_3)$, we apply the Riemann method to the second and third of equations (22), and obtain

$$\begin{aligned}\varphi(0, u_2, u_3) &= u_2^{-h}(u_2 - u_3)^h\varphi(0, u_2, 0) + u_3^{-h}(u_3 - u_2)^h\varphi(0, 0, u_3)\\ &\quad + \int_0^{u_2}\Phi(v_2, v_3;\ u_2, u_3)\varphi(0, v_2, 0)dv_2\\ &\quad + \int_0^{u_3}\Phi(v_3, v_2;\ u_3, u_2)\varphi(0, 0, v_3)dv_3,\\ \varphi(u_1, 0, u_3) &= u_3^{-h}(u_3 - u_1)^h\varphi(0, 0, u_3) + u_1^{-h}(u_1 - u_3)^h\varphi(u_1, 0, 0)\\ &\quad + \int_0^{u_3}\Phi(v_3, v_1;\ u_3, u_1)\varphi(0, 0, v_3)dv_3\\ &\quad + \int_0^{u_1}\Phi(v_1, v_3;\ u_1, u_3)\varphi(v_1, 0, 0)dv_1.\end{aligned}$$

Hence when these expressions are substituted in (33) we have the solution of (22) which is determined by the given initial values of $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$.

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